The Pressure Altimeter

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1 Introduction

The pressure altimeter works on the principle that the pressure of air varies in a known way with height. In electronic altimeter, the pressure is measured by piezo-resistive sensor. The output voltage is amplified and measured by A/D (Analog to Digital) converter. The air temperature is also measured. It is necessary due to strong temperature dependency of the pressure sensor. Besides, the pressure distribution of air also depends on the temperature.

This article deals with the dependency of atmospheric pressure on height. It is well known that the atmosphere has a vertical temperature lapse rate λ which is almost constant to about 11 km (according ISA - International Standard Atmosphere). An influence of the temperature lapse rate on pressure distribution is also analysed.

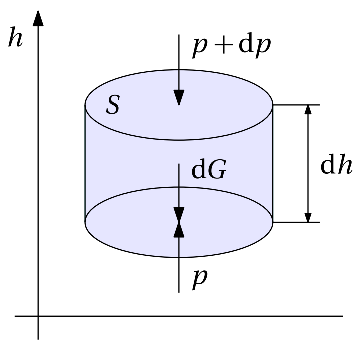

Figure 1: An infinitesimal element of atmosphere at equilibrium state.

Values used in the article:

p	–	Pressure, [Pa]
T	–	Absolute temperature, [K]
ρ	–	Density, [Kg/m^3]
g	–	Gravitational acceleration, [9.807 m/s^2]
R	–	Universal gas constant for air, [8.314 J/K/mol]
M	–	Molar mass of air, [$2.896 \cdot 10^{-2}$ kg/mol]
λ	–	Temperature lapse rate in atmosphere, [$-6.51 \cdot 10^{-3}$ K/m]

2 Derivation of Pressure Distribution

Consider an infinitesimal element of air in equilibrium. So, we can write

$$pS - \rho g S \mathrm{d}h - (p + \mathrm{d}p)S = 0, \tag{1}$$

where S is a surface of air element and $\mathrm{d}h$ is its height. We can arrange this equation and we obtain the first equation that describes the pressure

$$\mathrm{d}p = -\rho g \mathrm{d}h. \tag{2}$$

To obtain the second equation we can use the ***ideal gas law***

$$pV = nRT, \tag{3}$$

where n is the number of moles of gas that is equal to fraction of mass and molar mass

$$n = \frac{m}{M} = \frac{\rho V}{M}. \tag{4}$$

If we combine eqs. (3) and (4) we obtain $\rho = \frac{pM}{RT}$. We insert it to eq. (2) and obtain the final equation that describes the pressure distribution in atmosphere.

$$\boxed{\frac{\mathrm{d}p}{p} = -\frac{gM}{RT}\mathrm{d}h} \tag{5}$$

This is a differential equation. To obtain the pressure distribution, we need to solve it. We need to know how looks variables of this equation. It means, whether they are variable or constant. Because we will use our result only to small heights, lets consider that the gravitation field is homogenous. So, g is constant. Now lets solve eq. (5) with zero temperature lapse rate, and with non zero lapse rate. Then we compare both solutions.

2.1 Solution with Zero Temperature Lapse Rate

We will solve eq. (5) by straight integration from pressure p_1 at height h_1 to pressure p_2 at height h_2. Lets mark the height difference as $\Delta h = h_2 - h_1$.

$$\int_{p_1}^{p_2} \frac{\mathrm{d}p}{p} = -\frac{gM}{RT}\int_{h_1}^{h_2} \mathrm{d}h \tag{6}$$

The solution of this equation is

$$\ln\left(\frac{p_2}{p_1}\right) = -\frac{gM}{RT}\Delta h. \tag{7}$$

Now we express the pressure p_2 in the layer of Δh above the base layer with pressure p_1.

$$\boxed{p_2 = p_1 \exp\left(-\frac{gM}{RT}\Delta h\right)} \tag{8}$$

Similarly, we can express the height between layers with pressures p_1 and p_2.

$$\boxed{\Delta h = -\frac{RT}{gM}\ln\left(\frac{p_2}{p_1}\right)} \tag{9}$$

2.2 Solution with Non Zero Temperature Lapse Rate

We will proceed like in above solution. But we consider the non zero lapse rate. So, the temperature varies as

$$T(h) = T_1 + \lambda(h - h_1). \tag{10}$$

Now we will solve the eq. (5) by integration with substitution.

$$\int_{p_1}^{p_2} \frac{\mathrm{d}p}{p} = -\frac{gM}{R} \int_{h_1}^{h_2} \frac{1}{T(h)} \mathrm{d}h \tag{11}$$

$$\ln\left(\frac{p_2}{p_1}\right) = -\frac{gM}{R} \int_{h_1}^{h_2} \frac{1}{T_1 + \lambda(h - h_1)} \mathrm{d}h = \left| \begin{matrix} z = T_1 + \lambda(h - h_1) \\ \mathrm{d}z = \lambda \mathrm{d}h \end{matrix} \right| = -\frac{gM}{R} \int_{T_1}^{T_1 + \lambda\Delta h} \frac{1}{\lambda z} \mathrm{d}z$$

$$= -\frac{gM}{\lambda R} \ln\left(\frac{T_1 + \lambda\Delta h}{T_1}\right) \tag{12}$$

From this solution we can express the pressure p_2 and the height Δh.

$$\boxed{p_2 = p_1 \left(\frac{T_1}{T_1 + \lambda\Delta h}\right)^{\frac{gM}{\lambda R}}} \tag{13}$$

$$\boxed{\Delta h = \frac{T_1}{\lambda}\left[\exp\left[\frac{\lambda R}{gM} \ln\left(\frac{p_1}{p_2}\right)\right] - 1\right]} \tag{14}$$

The eq. (14) means, if we measure temperature T_1 and pressure p_1 in the base layer and pressure p_2 in the upper layer, we can calculate the distance Δh of the upper layer from the base layer.

Sometimes we measure temperature T_2 in the upper layer like in an aircraft model. In this case, we need to express the temperature function as

$$T(h) = T_2 - \lambda(h_2 - h). \tag{15}$$

By analogue procedure as in (12) we can obtain equation for Δh with T_2.

$$\ln\left(\frac{p_2}{p_1}\right) = -\frac{gM}{R} \int_{h_1}^{h_2} \frac{1}{T_2 - \lambda(h_2 - h)} \mathrm{d}h = \left| \begin{matrix} z = T_2 - \lambda(h_2 - h) \\ \mathrm{d}z = \lambda \mathrm{d}h \end{matrix} \right| = -\frac{gM}{R} \int_{T_2 - \lambda\Delta h}^{T_2} \frac{1}{\lambda z} \mathrm{d}z$$

$$= -\frac{gM}{\lambda R} \ln\left(\frac{T_2}{T_2 - \lambda\Delta h}\right) \tag{16}$$

$$\boxed{p_2 = p_1 \left(\frac{T_2}{T_2 - \lambda\Delta h}\right)^{-\frac{gM}{\lambda R}}} \tag{17}$$

$$\boxed{\Delta h = \frac{T_2}{\lambda}\left[1 - \exp\left[\frac{\lambda R}{gM} \ln\left(\frac{p_2}{p_1}\right)\right]\right]} \tag{18}$$

3 Pressure Calculation

Picture 2 shows computed height Δh that matches to pressure difference for temperature 15°C at base layer. The curve Ha is computed according eq. (14) which considers non zero temp. lapse rate. The curve Hb is computed according eq. (9) which considers zero lapse rate.

From computed samples is visible, the difference between both curves is a little at low heights, but for higher pressure difference is significant.

p1 [hPa]	p2 [hPa]	p1-p2 [hPa]	Ha [m]	Hb [m]	Ha-Hb [m]
1000	1000	0	0.0	0.0	0.0
1000	990	10	84.7	84.6	0.1
1000	970	30	256.1	255.3	0.7
1000	940	60	518.6	515.5	3.1
1000	890	110	971.6	960.9	10.7
1000	800	200	1841.8	1803.0	38.9
1000	600	400	4103.7	3907.2	196.5
1000	300	700	9070.1	8069.4	1000.8

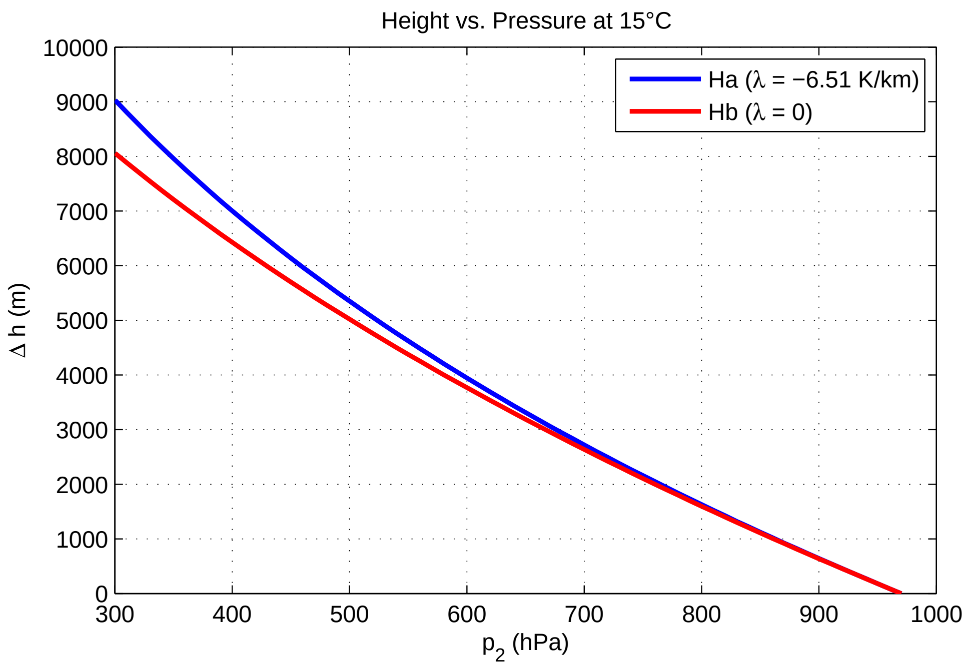

Figure 2: Height vs. pressure computed for non zero and zero temp. lapse rate.

Picture 3 shows computed height Δh according eq. (14) which considers non zero temp. lapse rate. There are several curves for different temperatures. It is visible, that at given height Δh the higher temperature means also higher pressure at that layer.

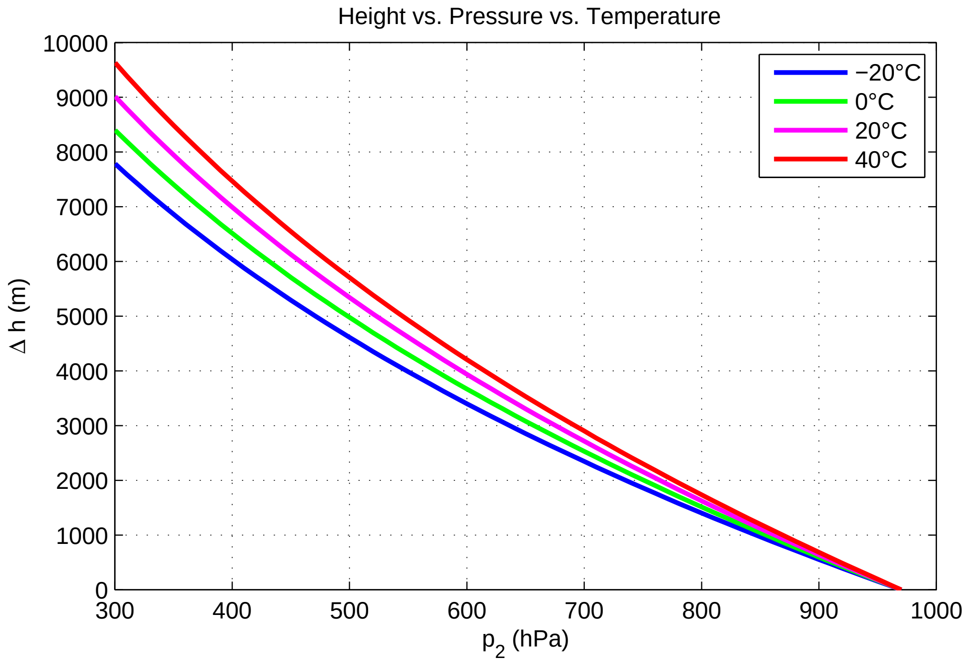

Figure 3: Height vs. pressure for different temperatures at base layer.

4 Conclusion

For altimeter that works within small pressure difference (small heights), the eq. (9) is fully suitable. At height 1 km, the error of this equation is only about 11 m. But for measuring upper heights, the eq. (14) or eq. (18) is necessary.

The best equation for aircraft model is the eq. (18). It enables to compute height Δh from pressure p_1 at the base layer, and from pressure p_2 and temperature T_2 at the upper layer.

$$\boxed{\Delta h = \frac{T_2}{\lambda}\left[1 - \exp\left[\frac{\lambda R}{gM}\ln\left(\frac{p_2}{p_1}\right)\right]\right]}$$